Properties and features of asymmetric partial devil's staircases deduced from piecewise linear maps

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(Received 8 August 2002; published 7 January 2003)

A piecewise linear map with one discontinuity is used to link together iterated map properties with the shape of the ensuing staircases. In the main part of the paper, a three-segment map is treated, with a horizontal middle segment next to the discontinuity and the development of partial and asymmetric staircases is demonstrated. In particular, a possible hierarchy of partiality, connected with the ratio of the length of the horizontal segment to the discontinuity jump, is obtained. The map is used for constructing staircases that imitate various experimental and numerical staircases that appear in the literature for excitable systems.

DOI: 10.1103/PhysRevE.67.016202 PACS number(s): 05.45.−a, 87.10.+e

I. INTRODUCTION

In many periodically driven oscillating systems plotting the winding number (or firing rate) r against some control parameter (like the period and the amplitude of the drive) yields a *complete devil's staircase*. In this plot, there exist plateaus at the levels of all rational values of $r \in [0,1]$. The length of these plateaus decreases with the denominator of r in its reduced form (see, e.g., Ref. [1]).

However, there exist systems, usually excitable, that exhibit one-sided highly asymmetric staircases, either partial or complete. Takahashi *et al.* [2] found, for a periodically stimulated giant squid axon, periods that were only of the types $\{1^n0\}$ and $\{100(10)^n\}$ (1 represents a forcing stimulus that triggers a pulse in the forced system, and 0 represents a stimulus that fails to do so). Thus, in this measurement only plateaus with $r=n/(n+1)$ and $r=m/(2m+1)$ $(m=n+1)$ were obtained in the devil's staircase. For a different range of parameters, aperiodic responses that made the staircase nonmonotonic have also been detected. In Ref. [3] the same group found an additional $\{10^3(10^2)^n\}$ sequence. Dolnik *et al.* [4] studied models of an excitable chemical system under a periodical pulse stimulation. For some of those models they obtained a complete devil's staircase, but for other models a one-sided partial staircase was obtained. It was shown that when a positively sloped part of a return map was artificially replaced by a horizontal segment, a complete staircase turned into a one-sided staircase, and vice versa. Sato and co-workers [5,6] obtained numerically the results similar to those of Takahashi *et al.* for an excitable Bonhoeffer Van der Pol system periodically forced by a train of pulses. In addition, for another range of parameters, they obtained a staircase of $\{10^n\}$ periods, with zones of bistability. Yasin *et al.* [7] studied the sinusoidally driven excitable Bonhoeffer Van der Pol system. A staircase with mostly $\{1^n0\}$, and $\{(10^n)(10^{n-1})^m\}$ (for few values of $n \geqslant 2$) plateaus was obtained. Huang *et al.* [8], measuring an optogalvanic circuit, found a $\{10^n\}$ staircase for one range of parameters, while for another parameter range gaps appeared between the various $\{10^n\}$ plateaus. These gaps were dominated by $\{10^n(10^{n-1})^m\}$ sequences. Fan *et al.* [9], studying a model for the concentration of calcium ions in nonexitable cells, obtained a staircase dominated by $\{10^n\}$ plateaus up to about $n=7$, and between every $\{10^n\}$ and $\{10^{n-1}\}$ plateau pairs they found a sequence of $\{(10^n)(10^{n-1})^m\}$ plateaus. (For $n=4$, a small $\{(10^5)(10^5 10^4)^m\}$ sequence appeared between the $\{10^5\}$ and the $\{10^5 10^4\}$ plateaus as well.) Soen *et al.* [10] obtained $\{1^n0\}$ and $\{100(10)^n\}$ phase locking in numerical solutions for the Morris-Lecar model. Coombes and Osbaldestin [11] studied the McKean model for two cases by constructing analytically a return map, which could be investigated. In one case the staircase consisted of only $\{10^n\}$ plateaus. In the second case the obtained staircase resembled those in Ref. [9], and had also chaotic regions. Ostborn *et al.* [12] studied models for the sinus node, and in particular a simplified model that treats the sinus node as a two-element system. The staircase obtained resembled the ones in [9,11].

For discrete maps, complete devil's staircases were obtained in maps that are monotonically increasing, except at one discontinuity point where they "jump" down, thus creating a two-branched map (one branch denoted by "0" while the other by "1"). This includes the sine circle map [13] where the mod 1 operation creates the discontinuity. Piecewise linear maps, consisting of two linear segments, one at each side of the discontinuity, were studied by Nagumo and Sato [14] for segments of equal slopes and by Yoshizawa *et al.* [15] for segments with different positive slopes. When one of the segments in such a piecewise linear map is not positively sloped, the staircase is no longer complete. Avrutin and Schanz [16] studied such a map where one segment was horizontal, and obtained a staircase containing only $\{1^n0\}$ plateaus. Rinzel and Troy [17] and LoFaro [18] studied maps with one segment that was negatively sloped. The staircases there contain $\{1^n0\}$ plateaus with bistability zones between each two neighboring plateaus.

The studies of maps with two linear segments mark the road that gives the connection between the general shape of a discrete map and the type of staircase that is thereby obtained. However, they fall short of explaining results like

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those in Refs. [2,3,9,10] where a much reacher variety of one-sided partial staircases can be found. As we presently show, this variety is intersting in itself and in addition shows a systematic order. The aim of the present work is to find a piecewise linear model that exhibits such staircases, and to study the general features of these maps.

In Sec. II, several general properties of the systems under study and some basic definitions are given. In Sec. III a geometrical approach is applied to the two-segment piecewise linear map emphasizing the role of the different parts of the map. This analysis also helps us to choose the parameters for the three-segment piecewise liner map discussed in Sec. IV. It is demonstrated that for some parameters, this map yields staircases that resemble the experimental and numerical results cited above. Finally, in Sec. V this comparison is discussed.

II. GENERAL PROPERTIES OF THE MODELS

All models we use are iterated maps of the form

$$x_{n+1}=f(x_n;a)=g(x_n)+a, \tag{1}$$

where $g(x)$ is a function of x with one discontinuity at which the value of the function drops, and a is a control parameter. $|df/dx|<1$ for all x, except at the discontinuity. We choose the discontinuity to be at $x=0$ with a jump of -1, thus

$$\lim_{x\to 0^-} f(x)=a,$$

$$\lim_{x\to 0^+} f(x)=f(0)=a-1, \tag{2}$$

as in Ref. [14]. All calculations are conducted for piecewise linear maps. The main features are, however, valid for more general maps having similar shapes.

The discontinuity divides the graph of the map into two regions:

$$R_0=(-\infty,0),$$

$$R_1=[0,+\infty). \tag{3}$$

For each orbit $\{x_i\}=\{x_1,x_2,\dots,x_n\}$, a binary sequence $\{J_i\}=\{J_1,J_2,\dots,J_n\}$ can be assigned, where

$$J_i=\begin{cases}0, & x_i\in R_0\\ 1, & x_i\in R_1.\end{cases} \tag{4}$$

With this sequence the *winding number* r can be calculated for a specific value of a. It is the number of 1's in the $\{J_i\}$ sequence divided by the sequence length, i.e.,

$$r=\lim_{n\to\infty}\frac{\sum_{i=1}^{n} J_i}{n}. \tag{5}$$

In actual calculations, though, n is finite. Here we use $n=300$. The orbit $\{x_i\}$ is started at an arbitrary point, beyond its transient beginning. There is a possibility of bistability, i.e., where two different winding numbers are obtained for the same value of a, for different initial values.

The condition $df/dx<1$ assures that for $a<0$ there exists only a single fixed point, which is located on R_0; for $a>1$, again only a single fixed point exists, but here it is located on R_1; and for $0<a<1$ no fixed points exist. Whenever a fixed point exists, it is stable since $|df/dx|<1$ for every $x\neq 0$. Thus, for $a<0$ there exists a basin of attraction, such that every orbit that starts in it converges to the fixed point, and so $r=0$, at least for some initial values. By similar reasonings, for $a>1$ there must be initial values for which $r=1$. Thus in order to obtain a devil's staircase, the range of a should include the range $[0,1]$, and in cases of bistability (see below) also values in its vicinity.

III. A TWO-SEGMENT MAP

A. The map

Since we are interested in asymmetric devil's staircases, we cannot use the symmetric Nagumo-Sato model [14]. The simplest possible model is the one in which R_0 and R_1 consist of only one linear segment each, but the two have different slopes. That is, we consider an iterated map $x_{n+1}=f(x_n)$ with

$$f(x)=\begin{cases}f_0(x)=b_0x+a, & x<0\\ f_1(x)=b_1x+a-1, & x\geqslant 0,\end{cases} \tag{6}$$

where $|b_i|<1$. We first discuss the case where both b_0 and b_1 are positive, and then the cases where one b is either 0 or negative. For each case we are interested in finding the borders, a_{n_L} from the left and a_{n_R} from the right, of the $\{1^n0\}$ plateaus, and $\tilde{a}_{n_L}$ and $\tilde{a}_{n_R}$ of the $\{10^n\}$ plateaus. We also try to find what other plateaus can be found in the gaps (when exist) between those plateaus.

B. The $\{1^n0\}$ and $\{10^n\}$ plateaus

Let us examine in detail the conditions for the appearance of an exclusive $\{1^n0\}$ period in the two-segment model, with $0<b_0<b_1$.

We begin by defining the two *crossing intervals* on R_1: The crossing-out interval O and the crossing-in one I (Fig. 1). These are the only intervals on R_1 which are involved in the crossing between R_1 and R_0. O is defined to be the interval that, when an orbit visits it, another iteration brings it to R_0, i.e.,

$$O:=\{x|x\geqslant 0,f(x)<0\}. \tag{7}$$

Hence $O=[O_L,O_R)$, where

$$O_L=0,$$

$$O_R=f_1^{-1}(0)=\frac{1-a}{b_1}. \tag{8}$$

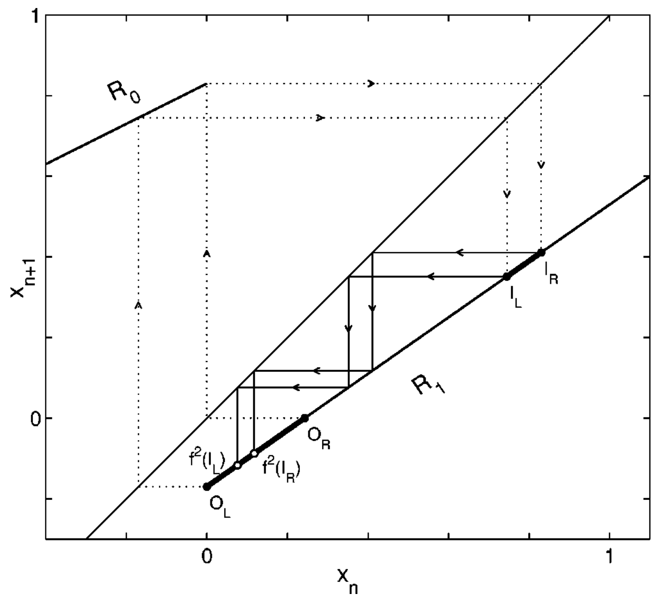

FIG. 1. The intervals $O=[O_L,O_R)$ and $I=[I_L,I_R)$ (thickened) of the map [Eq. (6)] with $a=0.83$, $b_0=0.5$, and $b_1=0.7$. For this case $f^2(I)\subset O$; therefore all orbits acquire a $\{1^3 0\}$ period after a short transient.

Before defining the segment I, let us find for what values of a a passage from R_1 to R_0 must be followed by an immediate return to R_1, that is, for what values of a an orbit (beyond the transient) cannot have two consecutive 0's? For this it is required that for every $x\in O$,

$$f^2(x)=f_0(f_1(x))>0. \tag{9}$$

Since $f_0(f_1(x))$ is monotonic, the condition is fulfilled when

$$f_0(f_1(0))>0, \tag{10}$$

or

$$a>\frac{b_0}{1+b_0}. \tag{11}$$

In a similar way, the condition for no two consecutive 1's is

$$f_1(f_0(0))<0; \tag{12}$$

thus,

$$a<\frac{1}{1+b_1}. \tag{13}$$

Since if $0<b_0,\ b_1<1$, it is always true that

$$\frac{b_0}{1+b_0}<\frac{1}{1+b_1}, \tag{14}$$

the two conditions (11) and (13) overlap. The range of the overlap is

$$\frac{b_0}{1+b_0}<a<\frac{1}{1+b_1}. \tag{15}$$

For this range (which always includes $a=\frac{1}{2}$), there can be neither two consecutive 1's nor two consecutive 0's. Therefore, Eq. (15) defines the minimal range of the $\{10\}$ plateau in the staircase. It can be shown that such a plateau also exists for the more general case, where the map is not piecewise linear but still obeys the conditions of Sec. II.

Now we can define for $a>b_0/(1+b_0)$ the interval I as the interval on R_1 to which orbits can return from R_0. Since, according to Eq. (11), for such values of a an orbit can visit R_0 only once,

$$I=f^2(O), \tag{16}$$

and the end points of this interval are

$$I_L=f^2(O_L)=f_0(f_1(0))=(b_0+1)a-b_0,$$

$$I_R=f^2(O_R)=f_0(0)=a. \tag{17}$$

We can now calculate the values of a for which the system has an exclusive $\{1^n 0\}$ period, regardless of the initial conditions, for $n>1$. A $\{1^n 0\}$ period means that after going from R_0 to R_1 (into interval I), the orbit visits R_1 for $n-1$ additional times before going back to R_0. A sufficient condition is

$$f_1^{n-1}(I)\subset O. \tag{18}$$

Figure 1 gives an example for such a case ($n=3$). Because $f_1(x)$ is a monotonous function, $f_1^k(I_L)<f_1^k(I_R)$. Therefore Eq. (18) implies

$$O_L<f_1^{n-1}(I_L)<f_1^{n-1}(I_R)<O_R. \tag{19}$$

While the calculations below are conducted specifically for the map of Eq. (6), the condition stated by Eq. (19) is valid for the general case of Sec. II.

Since

$$\begin{aligned} f_1^{n-1}(x)&=b_1 f_1^{n-2}(x)+a-1=\cdots\\ &=b_1^{n-1}x+\left(\sum_{i=0}^{n-2} b_1^i\right)(a-1)\\ &=b_1^{n-1}x+\frac{1-b_1^{n-1}}{1-b_1}(a-1), \end{aligned} \tag{20}$$

we have

$$f_1^{n-1}(I_R)=\frac{1-b_1^n}{1-b_1}a-\frac{1-b_1^{n-1}}{1-b_1} \tag{21}$$

and

$$\begin{aligned} f_1^{n-1}(I_L)=&\frac{1-b_1^n+b_0 b_1^{n-1}(1-b_1)}{1-b_1}a\\ &-\frac{1-b_1^{n-1}+b_0 b_1^{n-1}(1-b_1)}{1-b_1}. \end{aligned} \tag{22}$$

From Eq. (21) and the condition $f_1^{n-1}(I_R)<O_R$, we have

$$\frac{1-b_1^n}{1-b_1}a-\frac{1-b_1^{n-1}}{1-b_1}<\frac{1-a}{b_1}, \tag{23}$$

and from Eq. (22) and $O_L<f_1^{n-1}(I_L)$, we get

$$0<\frac{1-b_1^n+b_0b_1^{n-1}(1-b_1)}{1-b_1}a-\frac{1-b_1^{n-1}+b_0b_1^{n-1}(1-b_1)}{1-b_1}. \tag{24}$$

Thus, a sufficient condition for the appearance of an exclusive $\{1^n0\}$ period [i.e., a winding number $r=n/(n+1)$] is

$$\frac{1-b_1^{n-1}+b_0b_1^{n-1}(1-b_1)}{1-b_1^n+b_0b_1^{n-1}(1-b_1)}<a<\frac{1-b_1^n}{1-b_1^{n+1}}. \tag{25}$$

We shall denote those expression as $a^*_{n_L}$ and $a^*_{n_R}$. We prove below that $a^*_{n_L}=a_{n_L}$ and $a^*_{n_R}=a_{n_R}$.

In a similar way, or just by replacing $b_0\leftrightarrow b_1$ and $a\leftrightarrow 1-a$, the condition for an exclusive $\{10^n\}$ period $[r=1/(n+1)]$ is

$$\frac{b_0^n(1-b_0)}{1-b_0^{n+1}}<a<\frac{b_0^{n-1}(1-b_0)}{1-b_0^n+b_1b_0^{n-1}(1-b_0)}. \tag{26}$$

C. The gaps between the $\{1^n0\}$ plateaus

It can be seen from Eq. (25) that for every n, $a^*_{(n-1)_R}<a^*_{n_L}<a^*_{n_R}<a^*_{(n+1)_L}$. For $a\in(a^*_{n_R},a^*_{(n+1)_L})$, there is no value of n for which Eq. (18) is satisfied, and the staircase might contain plateaus for periods more complicated than $\{1^n0\}$. This range must exist because the transition from the situation where $f_1^{n-1}(I)\subset O$ to the situation where $f_1^{n-1}(I)$ is completely outside O and $f_1^n(I)\subset O$ instead, is gradual. During the transition, the point O_R divides the interval $f_1^{n-1}(I)$ into two parts, of which only the left one is in the interval O. The other part of $f_1^{n-1}(I)$ is the one that after another iteration falls into O.

We thus define two subintervals in O (see Fig. 2 for the $n=3$ case):

$$O_n:=f^{n-1}(I)\cap O=[f^{n-1}(I_L),O_R),$$

$$O_{n+1}:=f^n(I)\cap O=[O_L,f^n(I_R)). \tag{27}$$

For $a\in(a^*_{n_R},a^*_{(n+1)_L})$, whenever an orbit passes through the interval O (after the first passage), it must be through either O_n or O_{n+1}. When it does so through O_n, it contributes a (1^n0) subperiod to the period, and when it passes through O_{n+1}, it contributes a $(1^{n+1}0)$ subperiod. It is evident that as a is increased from $a^*_{n_R}$ to $a^*_{(n+1)_L}$, O_{n+1} expands, while O_n shrinks.

In order to see how the subintervals O_{n+1} and O_n are mapped back into themselves, it is useful to see how they are mapped into the interval $f^{n-1}(I)(=f^{n+1}(O))$ that is composed of O_n to the left of point O_R, and $f_1^{-1}(O_{n+1})$ to its right. As a is increased from $a^*_{n_R}$ to $a^*_{(n+1)_L}$, we can see three

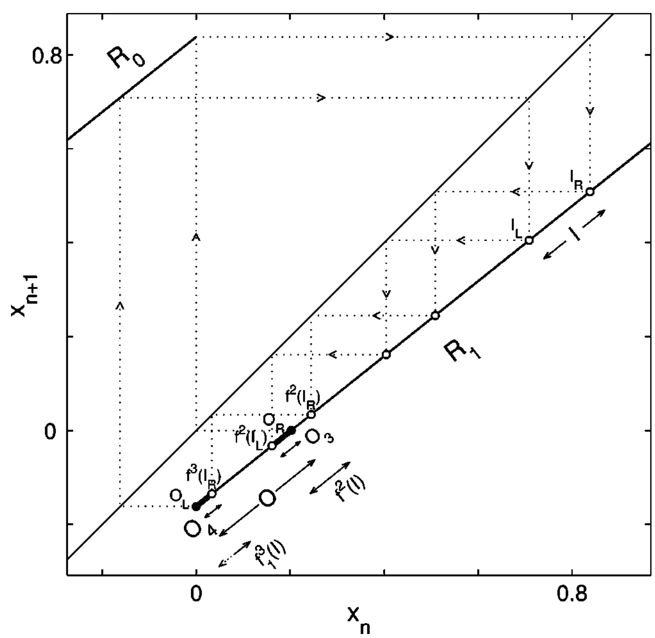

FIG. 2. The segments $O_4=[O_L,f^3(I_R))$ and $O_3=[f^2(I_L),O_R)$ (thickened) for $a=0.838$, $b_0=0.8$, and $b_1=0.8$. Orbits that leave R_1 through O_4 pass R_1 four times before leaving it, while those that leave through O_3 pass it three times. O_R divides $f^2(I)$ into two parts. To its left lies O_3, and to its right a part that after another iteration becomes O_4.

simultaneous processes: $f^{n+1}(O_{n+1})$ [which lies on the left side of $f^{n-1}(I)$] expands (with the expansion of O_{n+1} itself); $f^{n+1}(O_n)$ [which lies on the right side of $f^{n-1}(I)$] shrinks; and the point O_R crosses $f^{n-1}(I)$ from right to left. Thus, for small values of a, $O_R\in f^{n+1}(O_n)$; for intermediate values of a, O_R moves between $f^{n+1}(O_n)$ and $f^{n+1}(O_{n+1})$; and for yet higher values of a, O_R resides in $f^{n+1}(O_{n+1})$.

When O_R lies between $f^{n+1}(O_n)$ and $f^{n+1}(O_{n+1})$ (see Fig. 3 for the $n=2$ case),

$$f^{n+1}(O_{n+1})\subset O_n,$$

$$f^{n+2}(O_n)\subset O_{n+1}. \tag{28}$$

Thus, in this case, every $(1^{n+1}0)$ subperiod must be followed by a (1^n0) subperiod and vice versa, and the periodicity must be $\{(1^{n+1}0)(1^n0)\}$.

For $O_R\in f^{n+1}(O_n)$, there is a part of O_n which is mapped back into O_n (Fig. 4). In this case it is possible to have several consecutive (1^n0) subperiods. This allows periods like $\{(1^n0)^m(1^{n+1}0)\}$ and even more complicated ones like $\{[(1^n0)^m(1^{n+1}0)][(1^n0)^{m+1}(1^{n+1}0)]^l\}$ [as long as they do not contain two consecutive $(1^{n+1}0)$ subperiods]. However, there is always a limit to the number of consecutive (1^n0) subperiods in this parameter range. For the piecewise linear system, this can be seen by calculating the length l_m of the subsegment

$$O_n^m:=f^{n+1}(O_n^{m-1})\cap O_n,$$

$$O_n^0=O. \tag{29}$$

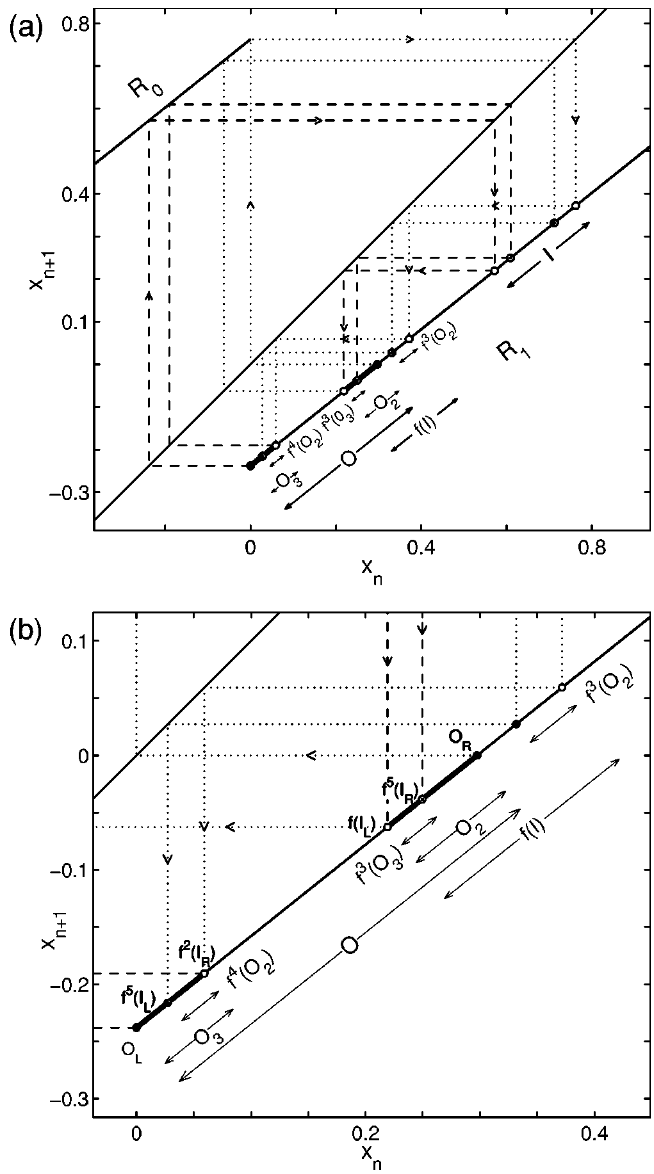

FIG. 3. (a) The segments O_3 and O_2 (thickened) for $a=0.762$ and $b_0=b_1=0.8$. Since for this case $f^3(O_3)\subset O_2$ and $f^4(O_2)\subset O_3$, there exists a $\{(1^3 0)(1^2 0)\}$ period. (b) Enlargement of the map around segment O.

Now, from Fig. 4,

$$l_m = q l_{m-1} - p = \frac{q^m(1+l_0-ql_0)-p}{1-q}, \tag{30}$$

where $q=b_0 b_1^n$, and p is the length of the subsegment

$$f^{n-1}(I) - O \cap f^{n-1}(I) = [O_R, f^{n-1}(I_R)]. \tag{31}$$

It is easy to see from Eq. (30) that for every value of a for which $O_R \in f^{n+1}(O_n)$, there is a final number $m_{max}(a)$ (which increases as $a \to a^*_{n_R}$ from above), so that after m_{max} consecutive visits in subsegment O_n, $l_{m_{max}+1}<0$, i.e., a

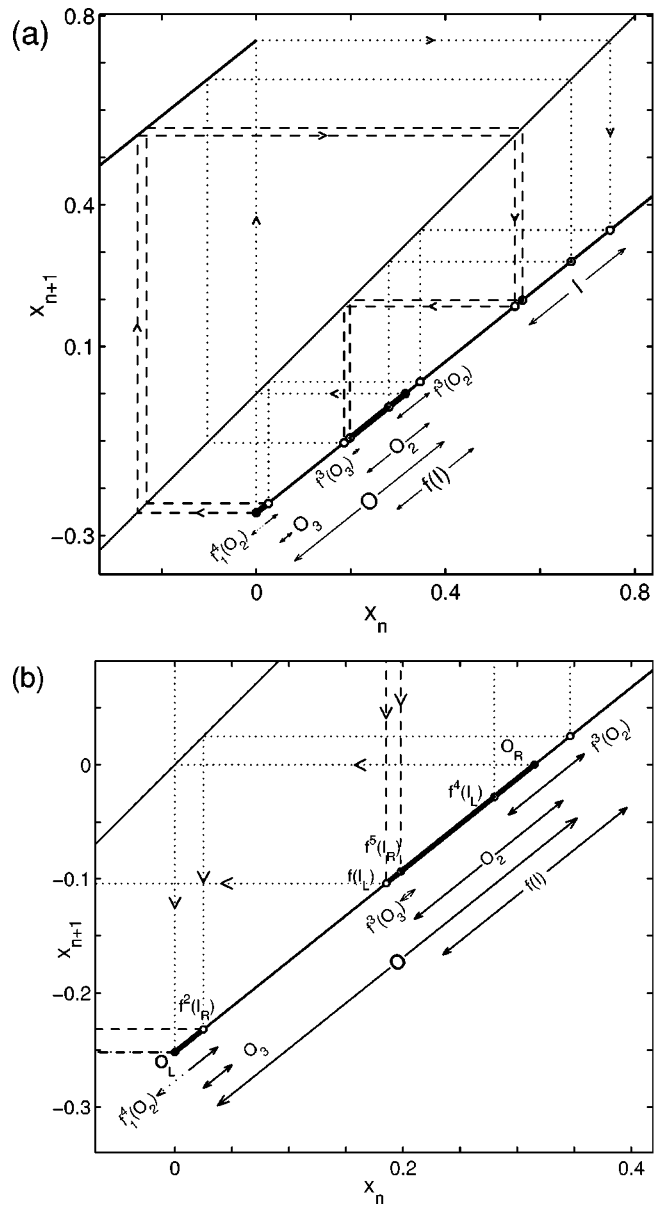

FIG. 4. (a) For $a=0.748$, and other parameters being the same as in Fig. 3, a part of $f^4(O_2)$ is in O_3, but another part falls out of O altogether. The part of O_2 for which f^4 falls outside O is that for which f^3 falls back into O_2. This allows several consecutive $(1^2 0)$ subperiods to appear between every $(1^3 0)$ one. (b) Enlargement of the map around segment O.

$(1^n 0)^{m_{max}}$ subperiod must be followed by a $(1^{n+1}0)$ subperiod. An important corollary is that for $a>a^*_{n_R}$, there can be no $\{1^n 0\}$ period, hence

$$a_{n_R} = a^*_{n_R}. \tag{32}$$

In a similar way, for $O_R \in f^{n+1}(O_{n+1})$, only consecutive $(1^{n+1}0)$ subperiods are allowed, and their number also has a final limit dependent on a, so we also have

$$a_{n_L} = a^*_{n_L}. \tag{33}$$

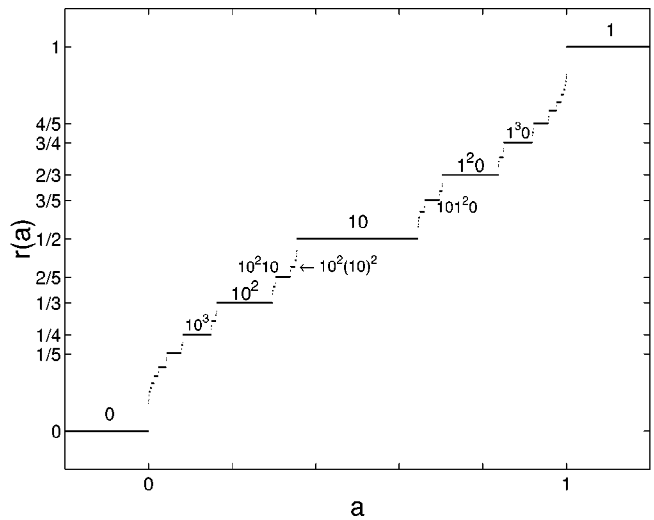

FIG. 5. Using Eq. (6) with $b_0 = b_1$ (0.55 in this case) yields a complete symmetric devil's staircase. This is exactly the Nagumo-Sato model.

For a general (not piecewise linear) map, Eqs. (32) and (33) can still be proved by using the same method using

$$q = \left[\max\left(\frac{df}{dx} \right) \right]^{n+1}. \tag{34}$$

D. Horizontal left segment

Now we can discuss the effect of changing b_0 and reducing it to 0. For $b_0 = b_1$, there is a complete symmetric devil's staircase (Fig. 5), as can be seen from Eqs. (25) and (26). This is the Nagumo-Sato model described in Ref. [14] (note that the slope in Ref. [14] is denoted by $1/b$).

Decreasing b_0 has several effects on the shape of the staircase. Its left side is usually affected by the "channel" between the diagonal and R_0. Decreasing b_0 opens up this channel. Thus, for a specific value of a, an orbit leaves R_0 after fewer visits there. This situation results in the expansion of the $\{10\}$ plateau towards the left, and the shrinkage of all other plateaus with $r < \frac{1}{2}$. For $b_0 = 0$ and $a > 0$, there cannot be two consecutive 0's at all. Thus, for $b_0 = 0$, all plateaus between $\{0\}$ and $\{10\}$ completely disappear. This result can be seen from Eq. (26), since for $b_0 = 0$ one gets $\tilde{a}_{n_L} = \tilde{a}_{n_R} = 0$ for every n.

The effect of decreasing b_0 on the right side of the staircase is less dramatic. As b_0 decreases, I shrinks while O is left unchanged. Since $f^{n-1}(I)$ shrinks too, it is more difficult to have $O_R \in f^{n-1}(I)$, and the gap between the $\{1^n0\}$ and the $\{1^{n+1}0\}$ plateaus shrinks. For $b_0 = 0$, I becomes a point. Therefore there is no range of a values for which $O_R \in f^{n-1}(I)$. Thus all gaps discussed above with all complicated plateaus between them disappear, and we are left with the $\{1^n0\}$ plateaus. This result can also be deduced from Eq. (25), since for $b_0 = 0$, $a_{n_R} = a_{(n+1)_L}$ for every n.

Figure 6 depicts a complete asymmetric devil's staircase obtained for $0 < b_0 < b_1 < 1$. Figure 7 depicts the partial staircase that is obtained for $b_0 = 0$ (as discussed in Ref. [16]).

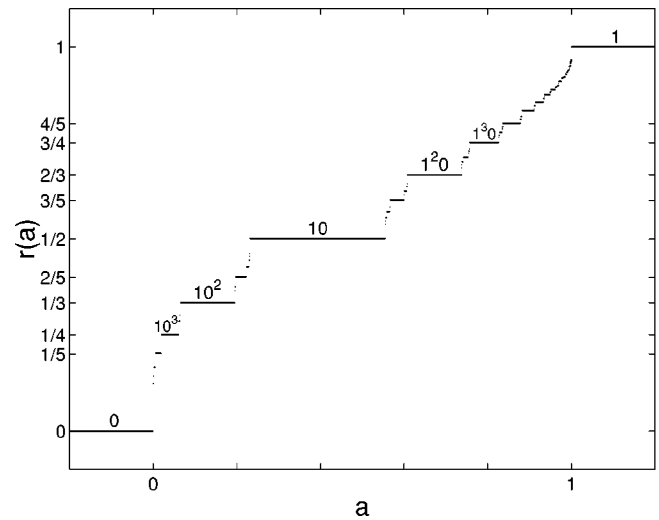

FIG. 6. Equation (6) with $b_0 = 0.3$ and $b_1 = 0.8$ yields a complete, but asymmetric, devil's staircase.

The gradual change of the staircase with b_0 can be seen in the upper part of Fig. 11.

E. Left segment with a negative slope

Consider the case where $b_0 < 0$, while $b_1 > 0$ (R_0 with a negative slope—studied by Rinzel and Troy [17] and by LoFaro [18]).

Here too, no consecutive 0's are possible for $a > 0$, because then, for all $x < 0$,

$$f_0(x) = b_0 x + a > a > 0. \tag{35}$$

Thus, there are no $\{10^n\}$ periods, or not even (10^n) subperiods, with $n > 1$.

Here O_L and O_R (see Fig. 8) are again given by Eq. (8). However, for the I interval, we have here the inverted relation [see Eq. (17)]:

$$I_L = f^2(O_R) = f_0(0) = a,$$

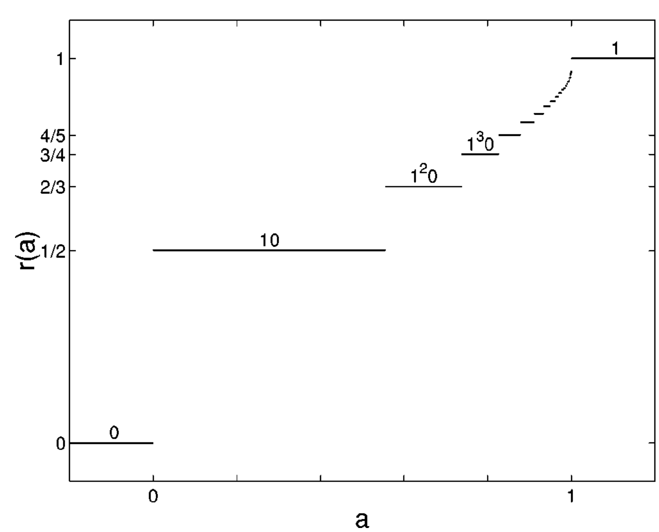

FIG. 7. Partial devil's staircase for $b_0 = 0$ and $b_1 = 0.8$.

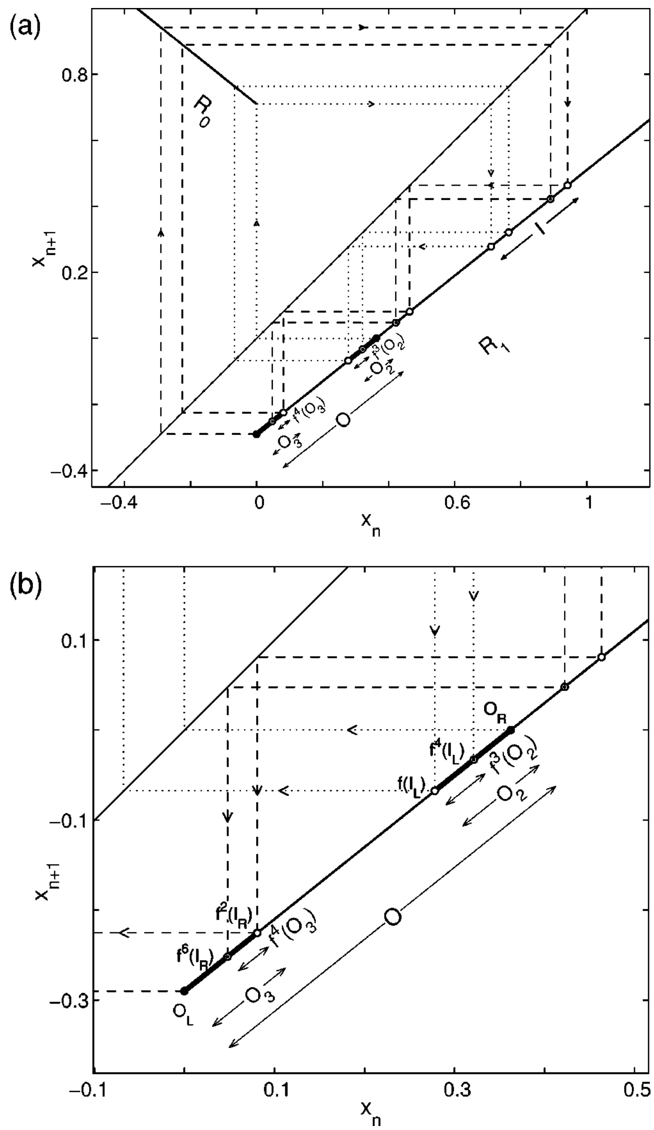

FIG. 8. (a) The segments O_3 and O_2 (thickened) for $a=0.71$, $b_0=-0.8$, and $b_1=0.8$. Since for the case shown, $f^4(O_3)\subset O_3$ and $f^3(O_2)\subset O_2$, the map with these parameters can have either a $\{1^3 0\}$ or a $\{1^2 0\}$ period, according to the initial condition. (b) Enlargement of the map around segment O.

$$I_R=f^2(O_L)=f_0(f_1(0))=(b_0+1)a-b_0. \tag{36}$$

Equation (19) is still the condition for an exclusive $\{1^n 0\}$ period, but with the expressions for I_R and I_L of Eq. (36), it yields

$$a^*_{n_L}=\frac{1-b_1^{n-1}}{1-b_1^n},$$

$$a^*_{n_R}=\frac{1-b_1^n+b_0 b_1^n(1-b_1)}{1-b_1^{n+1}+b_0 b_1^n(1-b_1)}. \tag{37}$$

Note that the expressions of $a^*_{n_L}$ and $a^*_{n_R}$ of Eq. (37) are identical to the expressions of a_{n-1_R} and a_{n+1_L}, respectively, obtained for the $b_0>0$ case.

Let us consider the a values of the gap between exclusive $\{1^n 0\}$ and $\{1^{n+1}0\}$ periods ($a^*_{n_R}<a<a^*_{n+1_L}$). As for the case where both b's are positive, the gap is characterized by

$$O_R\in f^{n-1}(I). \tag{38}$$

O_n and O_{n+1} are defined as in Eq. (27). We again look at the interval $f^{n-1}(I)$. Here too, $f^{n-1}(I)=f^{n+1}(O)$, but the order of points on O is revered. As in the positive b_0 case, the following three processes occur simultaneously when a is increased: the length of $f^{n+1}(O_{n+1})$ increases; the length of $f^{n+1}(O_n)$ decreases; and the point O_R crosses $f^{n-1}(I)$ from right to left. But here $f^{n+1}(O_{n+1})$ is on the right side of $f^{n-1}(I)$, while $f^{n+1}(O_n)$ is on its left. The crossing of $f^{n-1}(I)$ by O_R, the growth of $f^{n+1}(O_{n+1})$, and the shrinkage of $f^{n+1}(O_n)$ are all linear in a, but the fastest is the movement of O_R [which should cross $f^{n-1}(I)$ unlike the borders of $f^{n+1}(O_{n+1})$ and $f^{n+1}(O_n)$]. Hence O_R is always between $f^{n+1}(O_n)$ and $f^{n+1}(O_{n+1})$. Thus, for all values of a in the gap,

$$f^{n+1}(O_n)\subset O_n,$$

$$f^{n+2}(O_{n+1})\subset O_{n+1}, \tag{39}$$

and this gap is *bistable*. The number line is divided into two basins of attraction, one of the $\{1^{n+1}0\}$ period, and one of the $\{1^n 0\}$ period. Each of these basins is not simply connected. The former includes subsegment O_{n+1} (in particular, the point 0^+, which is easy to be used as an initial value in simulations); the latter includes O_n (and the point 0^-).

We thus see that

$$a_{n_L}=a^*_{n-1_R}<a^*_{n-1_R}=a_{n_L}<a^*_{n_R}=a_{n+1_L}<a_{n_R}=a^*_{n+1_L}, \tag{40}$$

where a_{n_L} and a_{n_R} are still given by Eq. (25).

We have seen above that for $a<0$ there is a stable fixed point on R_0. However, its basin of attraction does not necessarily include the whole x axis. Consider an interval $C=[C_L,C_R]$ on R_0 (Fig. 9), around the stable fixed point, defined by

$$C:=\{x|x<0;f(x)<0\}. \tag{41}$$

The borders of C are

$$C_L=f_0^{-1}(0)=-\frac{a}{b_0},$$

$$C_R=0. \tag{42}$$

Because

$$f(C_L)=0=C_R \tag{43}$$

and

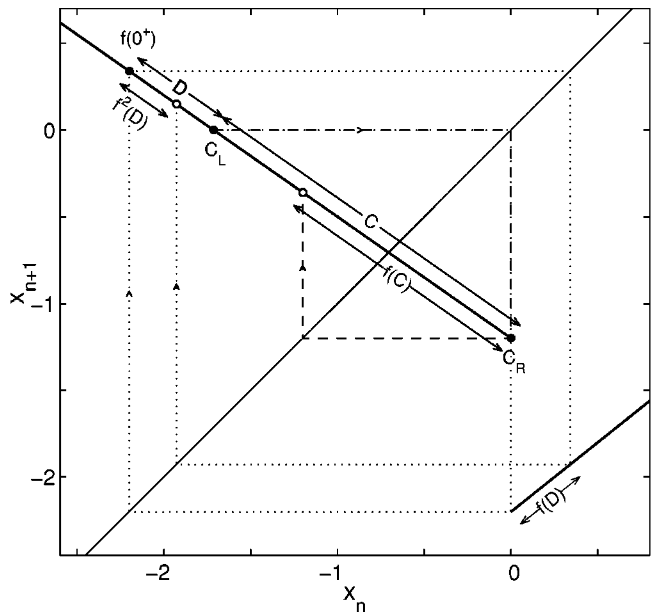

FIG. 9. Demonstration of a coexistence of a stable fixed point and a $\{10\}$ period: Interval C is a part of the basin of attraction of the fixed point. Interval $D=(f(0^+),C_L)$ is a part of the basin of the $\{10\}$ period. (The rest of the x axis is also divided between those two basins of attraction.) $a=-1.2$, $b_0=-0.7$, and $b_1=0.8$.

$$f(C_R)=a>C_L\,, \tag{44}$$

and since f is monotonic there, once an orbit is in C it stays there, converging into the fixed point.

In order to see whether the basin of attraction of the fixed point includes the whole x axis or not (coexistence with a $\{10\}$ period), consider the point $f(0^+)=a-1$. If $f(0^+)\in C$, then all orbits going over from R_1 to R_0 enter R_0 to the right of $f(0^+)$, and are thus in C. For such values of a, the basin of attraction of the stable fixed point is $(-\infty,\infty)$. If, however, $f(0^+)$ falls to the left of C, then $f^2(0^+)$ is in R_1 again, and it can be shown (see Fig. 9) that

$$f(0^+)<f^3(0^+)<C_L\,. \tag{45}$$

Thus, for the interval defined as

$$D=(f(0^+),C_L), \tag{46}$$

$$f^2(D)\subset D. \tag{47}$$

Now since

$$D\cap C=\emptyset\Rightarrow f(D)\subset R_1\,, \tag{48}$$

every orbit that passes through D must have a $\{10\}$ period.

We can see now that the condition for coexistence of $\{0\}$ and $\{10\}$ periods is

$$f(0^+)=a-1<-\frac{a}{b_0}=C_L\,, \tag{49}$$

or

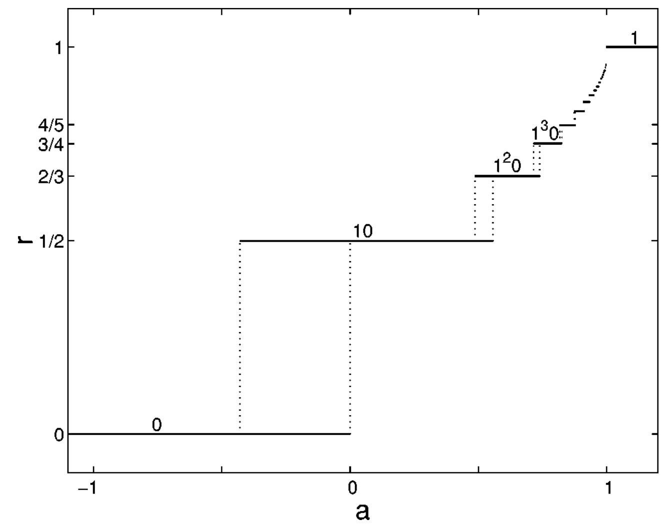

FIG. 10. Partial devil's staircase for $b_0=-0.3$ and $b_1=0.8$. The borders of the bistable zones are marked by dotted lines.

$$a>\frac{b_0}{1+b_0} \tag{50}$$

(recall that both a and b_0 are negative). So we see that also the expressions of the right border of the $\{0\}$ plateau and of the left border of the $\{10\}$ plateau stay unchanged as the sign of b_0 is flipped.

An example of a bistable partial staircase is given in Fig. 10, while the gradual growth of the bistability zones with b_0 is depicted in the lower part of Fig. 11.

IV. A THREE-SEGMENT MAP

In the preceding section, we have seen that the simple map used [Eq. (6)] enables either a complete staircase or a staircase with only $\{1^n0\}$ plateaus. Yet, it did demonstrate the link between a horizontal segment of the map and the partiality of the staircase. We now want to go one step fur-

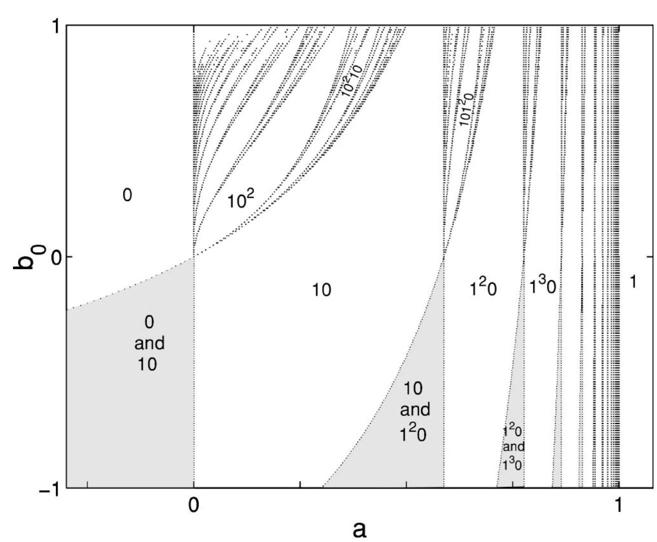

FIG. 11. The change of the staircase with b_0; $b_1=0.7$.

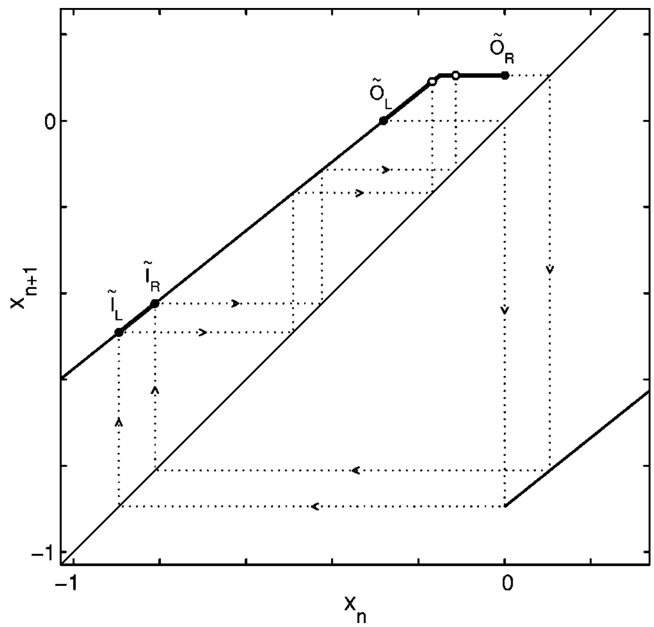

FIG. 12. A three-segment map with a horizontal segment for $a=0.105$, $b_0=0.8$, $b_1=0.8$, and $d=0.15$. $\tilde{O}=[\tilde{O}_L,\tilde{O}_R)$ and $\tilde{I}=[\tilde{I}_L,\tilde{I}_R)$ are thickened. For the case shown $[f^2(\tilde{I})\subset\tilde{O}]$, it therefore has a $\{10^3\}$ period.

ther, and to obtain a more complicated partial staircase. For example, the one with both $\{1^n0\}$ and $\{(10)^n10^2\}$ sequences of plateaus, as was reported by Takahashi and co-workers [2,3] and Soen *et al.* [10]. For this purpose, we try a map in which only part of R_0 is horizontal. Thus R_0 contains two linear segments. The segment next to the discontinuity is horizontal, and the other has a positive slope (Fig. 12). Formally, the iterated map function is

$$f(x)=\begin{cases} f_{0_\alpha}(x) & =b_0x+b_0d+a, \quad x\in S_{0_\alpha} \\ f_{0_\beta}(x) & =a, \quad x\in S_{0_\beta} \\ f_1(x) & =b_1x+a-1, \quad x\in S_1, \end{cases} \tag{51}$$

where

$$S_{0_\alpha}=(-\infty,-d), \quad S_{0_\beta}=[-d,0), \quad S_1=[0,+\infty),$$

and $0<b_0,\ b_1<1$.

We obviously have here a combination of the two previous behaviors (a R_0 with a positive slope, and a horizontal R_0). There are two extreme cases: for $d\to 0$, S_{0_β} disappears and the staircase must be complete; for $d\to\infty$, S_{0_α} disappears, and the staircase contains only $\{1^n0\}$ plateaus. The smallest value of x that an orbit can have after visiting R_1 is $f_1(0)=a-1$. Thus, for $d>1$, $f(0^+)\in S_{0_\beta}$ (for $0<a<1$, which is the significant range of a), so S_{0_α} is completely excluded from the dynamics, and the ensuing staircase is identical with the one that is obtained from a map with a totally horizontal R_0.

Consider, then, the case $d<1$. For $0<a<\frac{1}{2}$ (the left part of staircase), the diagonal is near R_0, therefore we should look closely at R_0.

Let us define for R_0 the intervals $\tilde{O}$ and $\tilde{I}$, in a manner similar to the definitions of O and I for R_1 (Fig. 12). $\tilde{O}=[\tilde{O}_L,\tilde{O}_R)$ where

$$\tilde{O}_L=f_{0_\alpha}^{-1}(0)=-d-\frac{a}{b_0},$$

$$\tilde{O}_R=0, \tag{52}$$

and $\tilde{I}=[\tilde{I}_L,\tilde{I}_R)$ where

$$\tilde{I}_L=f_1(0)=a-1,$$

$$\tilde{I}_R=f_1(f_{0_\beta}(0))=a(b_1+1)-1. \tag{53}$$

The existence of a horizontal segment S_{0_β} in the edge of R_0 excludes the possibility of an infinitesimally narrow channel between R_0 and the diagonal which, as in a tangent bifurcation, would allow an unlimited number of consecutive 0's between each crossover with R_1.

In order to find the maximal number of consecutive 0's that an orbit can have for a certain value of d, we consider the map for $a=0^+$. For this value of a (just above the bifurcation in which the stable fixed point on R_0 disappears), the channel between R_0 and the diagonal is the narrowest, and $\tilde{I}_L$ is furthest to the left. For $a=0^+$, $\tilde{I}_L=\tilde{I}_R=-1$, and therefore all the paths of the orbit R_0 must be identical to each other. Hence, at $a=0$ the jump in the staircase is from $\{0\}$ to a $\{10^n\}$ plateau (with a value of n yet to be found). A necessary condition for the existence of a $\{10^n\}$ period is that the $n-1$ iteration of $\tilde{I}_L$ $(=\tilde{I}_R)$ would be still on R_0, i.e.,

$$f_{0_\alpha}^{n-1}(-1;a=0)<0. \tag{54}$$

Substituting Eq. (54) in Eq. (51) shows that a $\{10^n\}$ period appears in the staircase for

$$d<\tilde{d}_n=b_0^{n-2}\frac{1-b_0}{1-b_0^{n-1}}. \tag{55}$$

(The case $\tilde{d}_2=1$ was discussed above.)

What plateaus other then $\{10^n\}$ can appear in the staircase, for a certain value of d? As for the two-segment map, we define subintervals $\tilde{O}_{n-1}$ and $\tilde{O}_n$ (Fig. 13) for values of a for which $\tilde{O}_L\in f^{n-2}(\tilde{I})$ (i.e., which correspond to the gap between the $\{10^n\}$ and the $\{10^{n-1}\}$ plateaus). Note that the $\tilde{O}$ interval consists of the whole S_{0_β} segment and a part of the S_{0_α} one, and that for

$$d>\frac{b_0^{2n-2}b_1(1-b_0)}{(1+b_0^{n-1}b_1)(1-b_0^n)}, \tag{56}$$

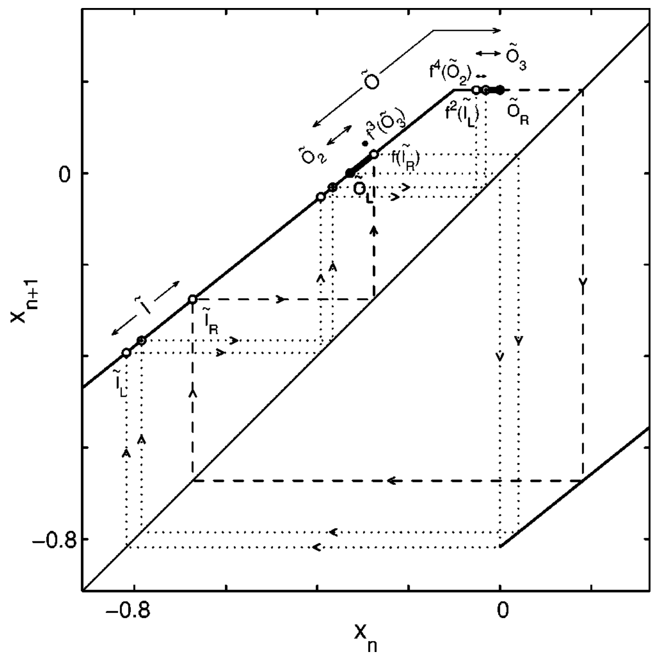

FIG. 13. The segments $\tilde{O}_2$ and $\tilde{O}_3$ (thickened) for $a=0.182$ $b_0=0.8$, $b_1=0.8$, and $d=0.1$. Note that here $\tilde{O}_3$ is mapped into one point.

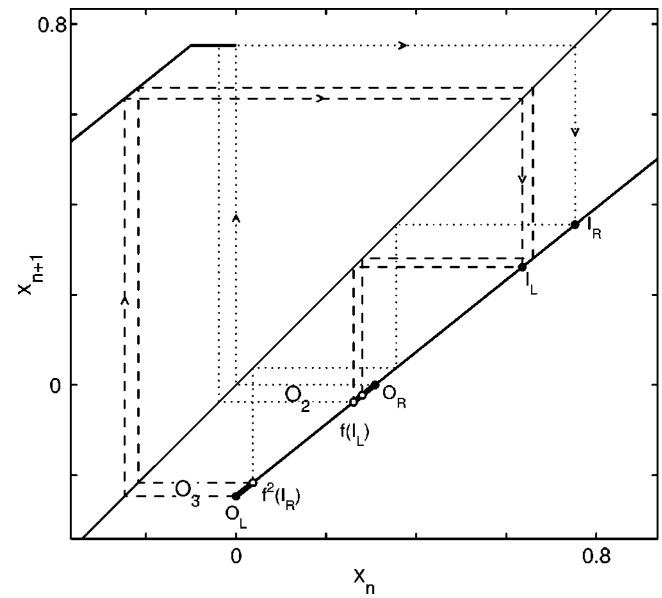

FIG. 14. The segments $O_3=[O_L, f^2(I_R))$ and $O_2=[f(I_L),O_R))$ (thickened) for $a=0.753$, $b_0=0.8$, $b_1=0.8$, and $d=0.1$. O_2 is mapped into one point after a visit in R_0.

for which

$$f^{n-1}(\tilde{I}_L;\tilde{a}_{n_R})>-d \tag{57}$$

is satisfied, the whole $\tilde{O}_n$ subinterval (that is always found on the right side of $\tilde{O}$) lies on the horizontal S_{0_β} segment. Thus, after passing through $\tilde{O}_n$, the orbit returns to R_0 at $\tilde{I}_R$ (Fig. 13 for $n=3$) regardless of the point at which it entered $\tilde{O}_n$. Thus every (10^n) subperiod must be followed by a (10^{n-1}) one [otherwise, every (10^n) subperiod must be followed by another identical (10^n) subperiod, which would bring us back to the $\{10^n\}$ period]. $\tilde{O}_{n-1}$, however, must have at least a part on the S_{0_α} segment. This allows successive visits in $\tilde{O}_{n-1}$, which are different from each other, and for which the last visit returns to $\tilde{O}_n$. Thus, there is no limitation on successive (10^{n-1}) subperiods. Hence, for $d<\tilde{d}_n$ there can be (and it turns out that there is) a sequence of $\{(10^n)(10^{n-1})^m\}$ $(m=1,\ldots,\infty)$ plateaus in the gap between $\{10^n\}$ and $\{10^{n-1}\}$. For values of d satisfying Eq. (56), this sequence fills the whole of this gap. Only values of d that are very small (compared to $\tilde{d}_n$) violate Eq. (56). For these values other periods begin to appear in this gap.

Let us now investigate how the right part of the devil's staircase changes with d. In order to do so we again look at the segments O and I on R_1 for $\frac{1}{2}<a<1$ (the right side of the staircase). The expressions for $O_L=0$, $O_R=f_1^{-1}(0)=(1-a)/b_1$ and $I_R=f(0^-)=a$, which are given by Eqs. (8) and (17) for the two-segment map stay unchanged, and are all independent of d. As we have seen above, the right end of each $\{1^n0\}$ plateau, a_{n_R}, is the value of a for which $f_1^{n-1}(I_R)=O_R$. Since I_R, O_R, and $f_1(x)$ are all independent of d, so is a_{n_R}. However, $I_L=f^2(0^+)$ does depend on d:

$$\begin{aligned} I_L&=f^2(0^+)=f(a-1)\\ &=\begin{cases} f_{0_\alpha}(a-1) &=b_0(a-1+d)+a, \quad -d\geqslant a-1\\ f_{0_\beta}(a-1) &=a=I_R, \qquad\qquad\quad -d<a-1. \end{cases} \end{aligned} \tag{58}$$

The left end of the $\{1^{n+1}0\}$ plateau, $a_{(n+1)_L}$, is the value of a for which $f_1^{n-1}(I_L)=O_R$. Therefore

$$-d<a_{n_R}-1\Rightarrow I_L=I_R\Rightarrow a_{n_R}=a_{(n+1)_L}, \tag{59}$$

that is, there is a jump from the $\{1^n0\}$ plateau directly to the $\{1^{n+1}0\}$ one. By substituting the expression of a_{n_R} from Eq. (25) into Eq. (59), we obtain

$$d_n=b_1^n\frac{1-b_1}{1-b_1^{n+1}}. \tag{60}$$

Thus for $d>d_n$, there is no gap between the $\{1^n0\}$ and the $\{1^{n+1}0\}$ plateaus. For $d<d_n$, there exist values of a for which the subintervals O_n and O_{n+1} can be defined in the O interval. Since $f(O_n)$ is on the right side of $f(O)$ (Fig. 14), its image might lie entierly in the horizontal part of the map, i.e.,

$$f(O_n)\subset S_{0_\beta}. \tag{61}$$

A necessary condition for this situation is

$$f_1^n(I_L;a_{n_R})>-d, \tag{62}$$

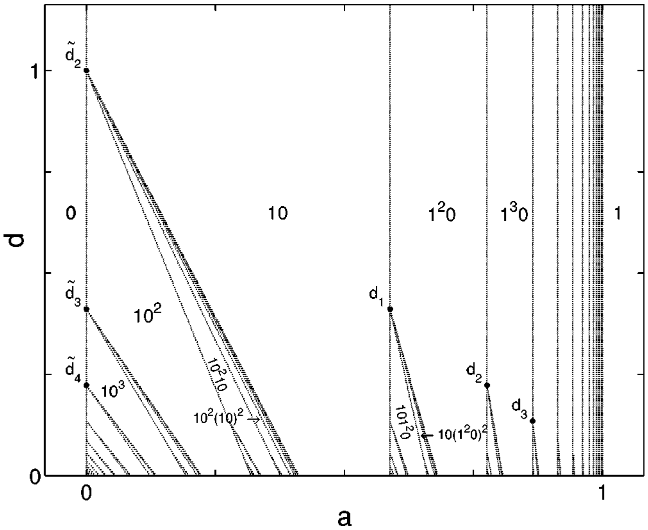

FIG. 15. The change of the staircase with d (the length of the horizontal segment S_{0_β}); $b_0=b_1=0.7$.

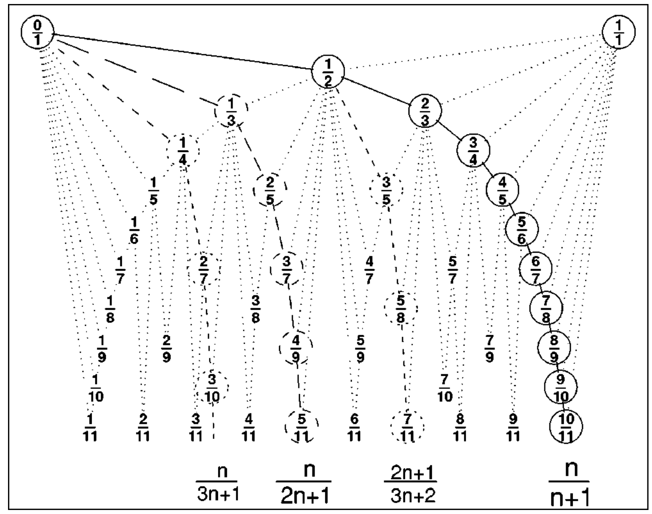

FIG. 16. The primary (full line), secondary (long-dashed), and two tertiary (short-dashed) branches of the Farey tree according to the order that is induced by the change of d in Eq. (51).

which is satisfied for

$$d > \frac{b_0 b_1^{2n}(1-b_1)}{(1-b_1^{n+1})(1+b_0 b_1^n)}. \tag{63}$$

For those values of d, the subinterval O_n maps to the point I_R, and later into $f^n(I_R)$ that is in O_{n+1}. Thus (in the gap) every $(1^n 0)$ subperiod must be followed by a $(1^{n+1}0)$ one, and there can be no period containing two consecutive $(1^n 0)$ subperiods. On the other hand, $f(O_{n+1})$ is on the left side of $f(O)$, therefore at least a part of it must lie in S_{0_α}, and there is no limit to consecutive $(1^{n+1}0)$ subperiods. Hence, for $d<d_n$, there can be (and actually there exists) a sequence of $\{(1^n 0)(1^{n+1}0)^m\}$ $(m=1,\ldots,\infty)$ plateaus in the gap between the $\{1^n 0\}$ and $\{1^{n+1}0\}$ plateaus. When in addition Eq. (63) is satisfied, this sequence fills the entire gap.

Figure 15 sums up the changes of the staircase with d, and reveals a hierarchy of partiality. For $d>1$ the staircase is constant, and contains only the $\{1^n 0\}$ plateaus. This range of d values is the widest, which puts this staircase in the top of the partiality hierarchy. The first change occurs at $d=1$, where a gap is being opened between the $\{0\}$ and the $\{10\}$ plateaus. This gap is filled with $\{(10^2)(10)^n\}$ plateaus. The range of d values here is also relatively large. A further decrease in d add a $\{(10^3)(10^2)^n\}$ sequence of plateaus between the $\{0\}$ and the $\{10^2\}$ plateaus, and a $\{(10)(1^2 0)^n\}$ sequence between the $\{10\}$ and the $\{1^2 0\}$ ones. The order of appearance of these two sequences depends on the ratio between b_0 and b_1. Equations (55) and (60) show that for $b_0=b_1$, $\tilde{d}_3=d_1$, and those sequences appears together. However, for $b_0>b_1$, the $\{(10^3)(10^2)^n\}$ sequence appears first, while for $b_1>b_0$ it is the $\{(10)(1^2 0)^n\}$ sequence that is the first to emerge.

With a further decrease of d, there appear the $\{(10^n)(10^{n-1})^m\}$ and $\{(1^n 0)(1^{n+1}0)^m\}$ sequences for $n>1$, which have been discussed above. Other sequences not previously discussed appear as well. The first one is a $\{(10)(101^2 0)^n\}$ sequence that appears between the $\{10\}$ and the $\{101^2 0\}$ plateaus. It starts to appear when $f(O_1)$ expands into the S_{0_α} segment (for values of a corresponding to the gap between the $\{10\}$ and the $\{1^2 0\}$ plateaus). As $d\to 0$, more and more periods are added until evidently a complete devil's staircase is achieved for $d=0^+$.

Generally speaking, the more complicated the partial staircase is, the narrower is the range of d values which yields it.

V. DISCUSSION

Asymmetric partial devil's staircases have appeared sporadically in the literature for more than a decade. In some of the cases they are explicitly identified, while for others the partiality of the staircase is not mentioned. But there has been no theoretical framework for partial devil's staircases, like the one existing for the complete ones. There does not even exist a "standard" map, playing the role that the sine circle map fulfills for the complete staircase.

In this paper, for the first time (as far as we know), such a standard is suggested [Eq. (51)]. The properties of the map were linked to the shape of the ensuing staircase. It has been shown that in this type (a map with a discontinuity), the long-term dynamics is determined only by the region around the discontinuity. Whenever this region is monotonically increasing (except at the discontinuity itself), the ensuing staircase is a complete one; and a horizontal (or negatively sloped) segment next to the discontinuity yields a partial staircase.

The link is not just qualitative, but semiquantitative. The essential feature of the map, which is responsible for the partiality of the staircase, is the ratio between the length of the horizontal (or almost horizontal) segment and the height of the discontinuity jump. This ratio is well defined for the piecewise linear map used here [d in Eq. (51)]. The definition is of course less accurate for more realistic maps, but is still important.

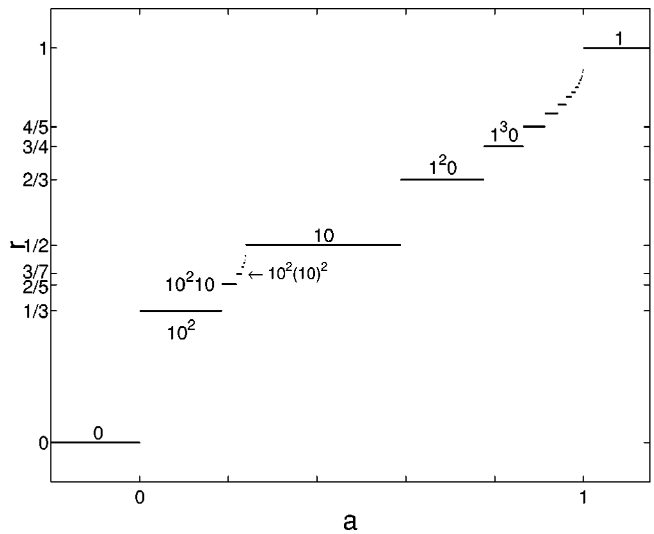

FIG. 17. Staircase with both $\{1^n 0\}$ and $\{(1^2 0)(10)^n\}$ sequences of plateaus for the three-segment map with $d=0.42$. Other parameters are as in Fig. 15.

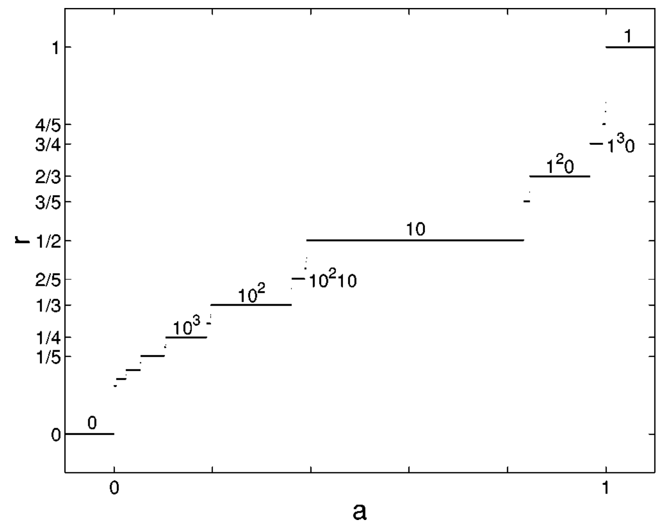

FIG. 18. Staircase for $b_0=0.7$, $b_1=0.2$, and $d=0.05$. Since $d_2<\tilde{d}_9<d<\tilde{d}_8<d_1$, there is a $\{(10^8)(10^7)^n\}$ sequence on the left side of the staircase, whereas on the right side the only sequence other than the $\{1^n 0\}$ is a small $\{(1^2 0)^m 10\}$ sequence.

The use of this ratio induces a hierarchy in the emerging branches of the Farey tree. It can be seen from Fig. 16 that all the sequences of plateaus that appear in the devil's staircase of Eq. (51) as d is decreased, correspond to branches of the Farey tree which incline to the right. The primary branch is the $n/(n+1)$ branch (for the $\{1^n 0\}$ sequence of plateaus). It grows from the $\frac{0}{1}$ leaf and is "pulled" by the $\frac{1}{1}$ leaf. The secondary branch is the $n/(2n+1)$ (for the $\{(10)^n 10^2\}$ plateaus). It grows from the first leaf in the primary branch ($\frac{0}{1}$) and is pulled by the second leaf ($\frac{1}{2}$). There are two tertiary branches. The $n/(3n+1)$ (for the $\{(10^2)^n 10^3\}$ plateaus) that grows from the first leaf of the secondary branch, and the $(2n+1)/(3n+2)$ (for the $\{(1^2 0)^n 10\}$ plateaus) that grows from the second leaf of the primary branch. There are four quarternary branches growing in a similar way, and so forth. For $b_0=b_1$ in Eq. (51), both tertiary branches appear together when d is reduced, followed by all the quaternary branches, etc. But for $b_0>b_1$, the whole tree "leans" more to the $\frac{0}{1}$ side, i.e., the left branches in each level of the hierarchy appear first. Left branches from a lower level may sometimes appear even before right branches from an upper level in the hierarchy.

This hierarchy enables a successful "imitation" by the piecewise linear map of staircases that appear both in experiments and in numerical calculations of ordinary differential equations. The simplest staircase, other than the trivial ones that contain only $\{1^n 0\}$ (or $\{10^n\}$ when the horizontal segment is to the right of the discontinuity), is of the type that was obtained by Takahashi *et al.* in Ref. [2], where in addition to the $\{1^n 0\}$ sequence of plateaus there also appears a $\{(10)^n 10^2\}$ sequence. In order to imitate this staircase with the map of Eq. (51), a suitable value of d can always be chosen: $\max(\tilde{d}_3,d_1)<d<\tilde{d}_2$ (as can be seen in Fig. 17). A staircase, where in addition to these plateaus there appears also a $\{(10^2)^n 10^3\}$ sequence like in Ref. [3], can also be obtained. As we can see from Eqs. (55) and (60) and Fig. 15, in order to avoid the appearance of an additional $\{(1^2 0)^n 10\}$ sequence, b_1 should be smaller than b_0, i.e., R_0 (in its nonhorizontal segment) should be steeper than R_1. In some papers [9,7,11] there exist partial staircases that contain many $\{(10^n)(10^{n-1})^m\}$ sequences, and relatively narrow $\{(10)(1^2 0)^n\}$ plateaus. This kind of staircase can be achieved (Fig. 18) with $b_0 \gg b_1$, which is needed in order to obtain $\tilde{d}_n>d_1$ for a relatively high value of n. This result is consistent with maps obtained for excitable models [11,19], where the slope of R_0 is almost 1, except for a small segment near the discontinuity where R_0 curls downwards, and R_1 monotonically, but moderately, increases.

The piecewise linear map is obviously only a caricature of real maps, for which R_0 curls down near the discontinuity, but it does explain the partiality phenomenon quite well. It fails, however, to recapture finer details like "period doubling" and chaos (the later, since $|df/dx|<1$). This is probably why we have not obtained any "noisy" nonmonotonic staircase, like those found in several papers [2,3,11,12]. These are the subject matter for future publications.

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